Propagation Constants for TE and TM Surface Waves on an Anisotropic Dielectric Cylinder*

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Summary—Maxwell's equations for wave propagation in a cylindrical anisotropic dielectric rod have been solved for various values of the longitudinal and transverse dielectric constants with the help of an IBM 7090 computer. The solutions are limited to modes having no rotational dependence about the direction of propagation. Families of curves for various ratios of longitudinal to transverse dielectric constants are given, showing the relationship between the guided wavelength and the diameter of the rod. Equations for the cutoff and asymptotic behavior are also given.

Introduction

With the recent development of the optical maser and the attendant high-power, monochromatic, coherent radiation, attention has been focused upon nonlinear interactions in solid media which heretofore have been almost impossible to produce. It is well known that the interaction of electromagnetic waves in a solid is produced by nonlinearities in the medium, and that the symmetry properties of the medium govern to a large extent the magnitude of the interaction, the greatest effect occurring when the medium lacks a center of inversion symmetry.[1,2] For many experiments, this requirement leads to a crystal which is anisotropic in many of its properties. In addition, it is usually necessary to match the velocities of the waves involved to produce as great an interaction length as possible. One experimental proposal is the parametric generation of microwaves by an optical maser beam through nonlinear interactions in a crystalline solid.[3,4] This requires the phase-matching of microwaves to optical waves in a medium which has an anisotropic dielectric constant. This may be accomplished conveniently by utilizing the high-phase velocity (relative to that in the infinite dielectric) of a surface wave on a cylindrical dielectric rod. Since little work has appeared in the literature concerning the propagation of microwaves on an anisotropic dielectric has prompted this calculation.

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[1] P. A. Franken and J. F. Ward, "Optical harmonics and nonlinear phenomena," *Revs. Modern Phys.*, vol. 35, pp. 23–39; January, 1963.

[2] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, "Interactions between light waves in a nonlinear dielectric," *Phys. Rev.*, vol. 127, pp. 1918–1939; September, 1962.

[3] R. H. Kingston, Parametric amplification and oscillation at optical frequencies, Proc. IRE (*Correspondence*), vol. 50, p. 472; April, 1962.

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The solution of Maxwell's equations for a cylindrical system is straightforward both for the case of isotropic and anisotropic dielectric constants. However, in order to obtain information other than the cutoff condition and asymptotic behavior of the waves, two transcendental equations must be solved simultaneously. Graphically, this is a laborious process and precludes much more than a point solution. An IBM 7090 computer has been programmed to solve certain of these equations, enabling us to obtain the propagation information rapidly for any value of dielectric constant. Only modes independent of the angle of rotation about the rod axis will be considered, since only for such modes is separation into pure transverse magnetic waves (TM) or transverse electric waves (TE) possible.

TM Solutions

It is assumed that the dielectric rod and surrounding medium have permeability μ and that both are lossless. The surrounding medium is assumed to have an isotropic dielectric constant ϵ, and the rod, anisotropic dielectric constants, ϵ_T in the transverse direction (radial) and ϵ_L in the longitudinal direction (along the rod axis). This is treated by writing

$$\boldsymbol{\varepsilon} E = \epsilon_T E_T + \epsilon_L E_L = D$$

and by imposing the proper cylindrical boundary conditions on Maxwell's equations.

For a transverse magnetic (TM) wave, two equations are found which determine the propagation constants

$$xJ_0(x)/J_0'(x) = K_L y H_0(y)/H_0'(y) \tag{1}$$

$$x^2 + (K_L/K_T)(y/i)^2 = (\pi d/\lambda_0)^2(K_L - K_L/K_T) \tag{2}$$

where $x = \alpha_1 r$ and $y = \alpha_2 r$

$J_0(x)$ = Bessel function of zeroth order
$J_0'(x)$ = Derivative of zeroth order Bessel function
$H_0(y)$ = Zeroth order Hankel function of the first kind
$H_0'(y)$ = Derivative of zeroth order Hankel function of the first kind
r = Cylinder radius
d = Cylinder diameter
λ_0 = Wavelength of the unguided wave
$K_L = \epsilon_L/\epsilon$ = Ratio of longitudinal dielectric constant of the rod to the surrounding dielectric
$K_T = \epsilon_T/\epsilon$ = Ratio of transverse dielectric constant of the rod to the surrounding dielectric.

Upon setting $K_L/K_T=1$, (1) and (2) become the determining equations for the case of an isotropic dielectric rod. See, for example, Kiely[5] or Collin.[6]

The equations of the TM waves are:

inside the rod,

$$H_z = H_\rho = E_\phi = 0$$
$$E_\rho = iA[K_L/K_T][(\lambda/\lambda_0)^2K_L - K_L/K_T]^{-1/2}J_1(\alpha_1\rho)$$
$$E_z = AJ_0(\alpha_1\rho)$$
$$H_\phi = iAK_L[\epsilon/\mu]^{1/2}[(\lambda/\lambda_0)^2K_L - K_L/K_T]^{-1/2}J_1(\alpha_1\rho)$$

where

$$\alpha_1 = \pi(1/r)(d/\lambda_0)[K_L - (K_L/K_T)(\lambda_0/\lambda)^2]^{1/2};$$

outside the rod,

$$H_z = H_\rho = E_\phi = 0$$
$$E_\rho = iC[(\lambda/\lambda_0)^2 - 1]^{-1/2}H_1(\alpha_2\rho)$$
$$E_z = CH_0(\alpha_2\rho)$$
$$H_\phi = iC[\epsilon/\mu]^{1/2}[(\lambda/\lambda_0)^2 - 1]^{-1/2}H_1(\alpha_2\rho)$$

where

$$\alpha_2 = \pi(1/r)(d/\lambda_0)[1 - (\lambda_0/\lambda)^2]^{1/2}. \quad (3)$$

λ = Wavelength of the guided wave
$J_1(\alpha_1\rho)$ = First-order Bessel function
$H_1(\alpha_2\rho)$ = First-order Hankel function of the first kind.

A or C is an arbitrary constant, but the ratio of the two is determined by the equation $A/C = H_0(y)/J_0(x)$.

Because of the oscillatory nature of the Bessel function, (1) has an infinite number of solutions. These correspond to the dominant mode (*i.e.*, the mode with the lowest cutoff frequency) and all the higher-order modes. We will concern ourselves here with the dominant mode whose electric field configuration appears as shown in Fig. 1.

A convenient way of expressing the propagation information is to plot λ/λ_0 as a function of d/λ_0 by combining (1) and (2) with

$$(\lambda/\lambda_0)^2 = [1 - y^2(\lambda_0/\pi d)^2]^{-1} \quad (4)$$

which is a restatement of (3). Eqs. (1), (2), and (4) have been solved in this manner by an IBM 7090 computer. In order to get full utility from the information, a value of K_L/K_T is given on each graph and a family of curves is produced by then plotting various values of K_L, such that reasonably accurate interpolation can be carried out. The graphs are shown in Figs. 2–6.

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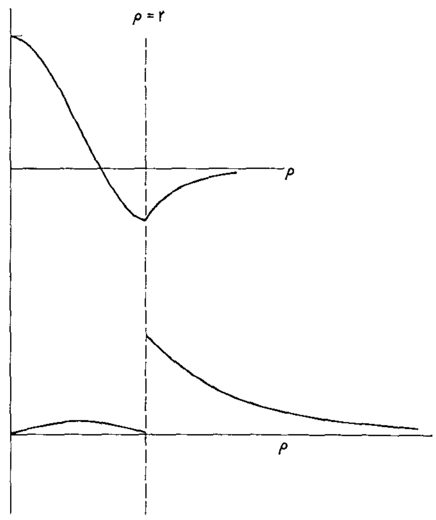

Fig. 1—Transverse rod section, dominant TM mode, $K_L=20$, $K_L/K_T=0.5$, and $\lambda/\lambda=0.67$.

TM Asymptotes and Cutoff

Two limiting equations can be written algebraically, the first of which is

$$\lim_{d/\lambda_0\to\infty} \lambda/\lambda_0 = [K_T]^{-1/2}. \quad (5)$$

The second, which gives the so-called cutoff condition, is

$$[d/\lambda_0]^{\text{Dominant Mode}}_{\text{C.O.}} = 2.405[1/\pi][K_L - K_L/K_T]^{-1/2}. \quad (6)$$

This is the minimum value of d/λ_0 for which bounded solutions exist and it occurs at $\lambda/\lambda_0 \geq 1$ when the argument of the Hankel function becomes real, whereupon the function itself becomes oscillatory.

In (6) the value 2.405 is the argument for the first zero of $J_0(x)$. The cutoff condition for the next highest mode is given by replacing 2.405 with the value of the argument which gives the next zero in $J_0(x)$. Cutoff for successively higher modes is obtained by an analogous process. The higher-order modes have the same limiting value of λ/λ_0 as shown in (5). It should be noted in the preceding that on setting $K_L/K_T=1$ the equations and propagation constants for TM waves in an isotropic rod are generated.

TE Solutions

By the nature of the TE waves, only K_T has any effect on their propagation; therefore, one need only substitute K_T for the dielectric constant in the solutions for the isotropic dielectric rod. The isotropic case has been treated for TM, TE and hybrid modes by Kiely.[1] Since our program also gives the TE solutions, a graph of the propagation constants for TE waves in an isotropic (anisotropic) dielectric rod has been included. This is shown in Fig. 7.

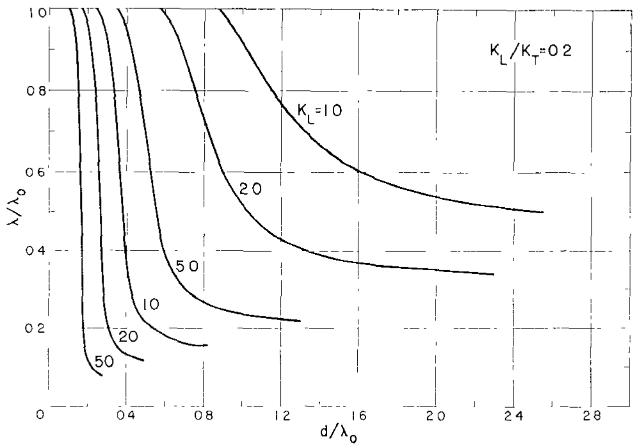

Fig. 2—Computer solutions for dominant TM wave propagation in an anisotropic dielectric rod.

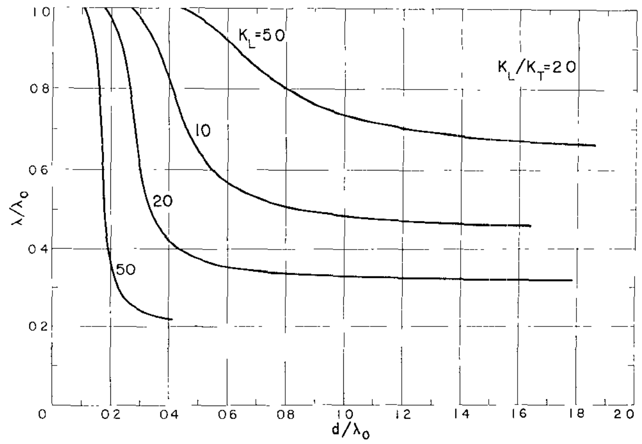

Fig. 5—Computer solutions for dominant TM wave propagation in an anisotropic dielectric rod.

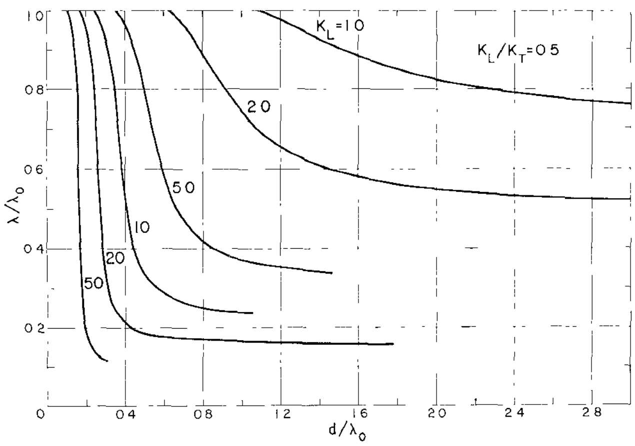

Fig. 3—Computer solutions for dominant TM wave propagation in an anisotropic dielectric rod.

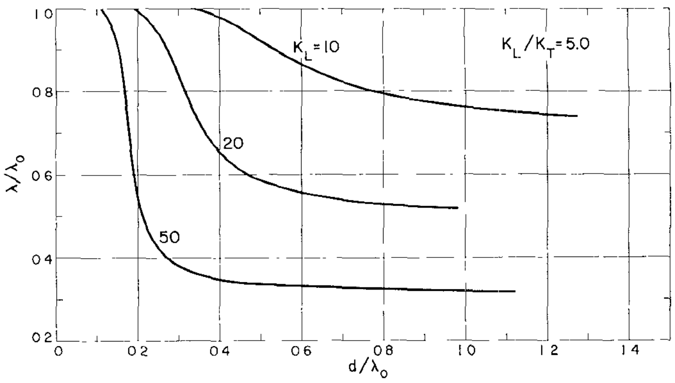

Fig. 6—Computer solutions for dominant TM wave propagation in an anisotropic dielectric rod.

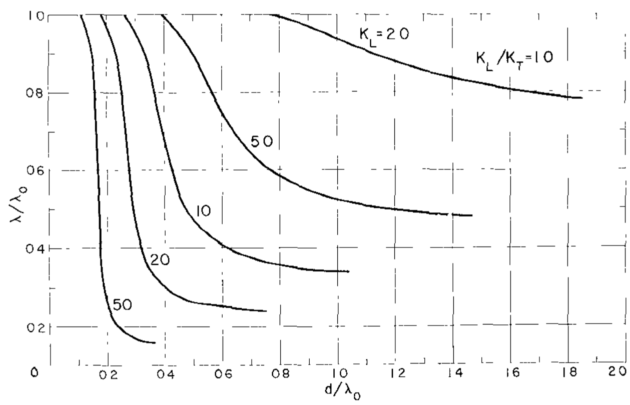

Fig. 4—Computer solutions for dominant TM wave propagation in an anisotropic dielectric rod.

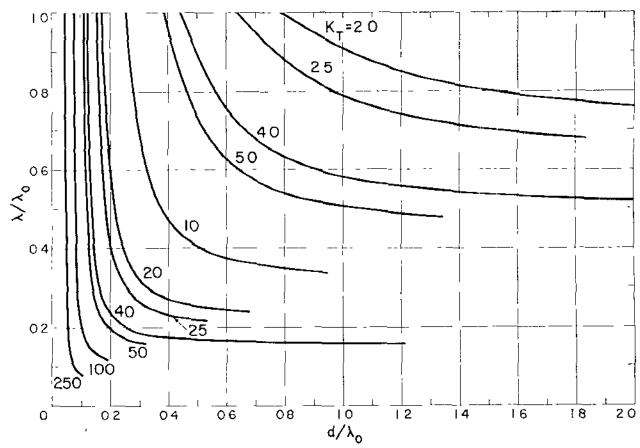

Fig. 7—Computer solutions for dominant TE wave propagation in an isotropic or anisotropic dielectric rod.

Conclusion

Unlike hybrid modes, the pure TM and TE modes have a minimum value of d/λ_0 for which they are bound to the rod. For values of d/λ_0 smaller than this, the modes are neither bound to the rod nor will propagate independently of it, hence are effectively cut off. This shows on the graphs as a nonzero slope for the curves at $\lambda/\lambda_0 = 1$, the respective values of d/λ_0 being given by (6) for the TM modes and by

$$[d/\lambda_0]_{\substack{\text{Dominant Mode}\\ \text{C.O.}}} = 2.405[1/\pi][K_T - 1]^{-1/2} \quad (7)$$

for the TE modes.

The asymptotic value of λ/λ_0 for large d/λ_0 is given by (5) for both TM and TE modes and is the value of λ/λ_0 that the modes would have in a dielectric medium of infinite extent.

It can be seen that any value of λ/λ_0 between 1 and $[K_T]^{-1/2}$ may be selected by proper choice of d/λ_0; however, as d/λ_0 becomes smaller, more of the energy of the wave is propagated outside the rod. In general, an increase in dielectric constant has the opposite effect of binding the wave more tightly to the rod. Since the microwave index of refraction may be varied from 1 to $[K_T]^{1/2}$, which is generally higher for a given dielectric than the respective optical index of refraction, one can almost always effect a match of velocities in an optical-microwave type experiment.

Acknowledgment

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Further Formulas for Calculating Approximate Values of the Zeros of Certain Combinations of Bessel Functions*

Introduction

In a recent letter Gunston[1] has presented a wonderfully simple approximate formula for the smallest z zero of the Bessel function equation

$$J_p(z)N_p(kz) - J_p(kz)N_p(z) = 0 \qquad (1)$$

where J_p and N_p are, respectively, the Bessel functions of the first and second kinds of real-order p. This communication is intended to draw attention to the existence of similar approximate formulas for both the larger z zeros of (1) and the roots of the equally-important companion equation

$$J_p'(z)N_p'(kz) - J_p'(kz)N_p'(z) = 0 \qquad (2)$$

where $'$ indicates differentiation.

Background

In the usual physical cases of interest the parameter p is an arbitrary real number while k is generally positive. It is known that under these conditions the zeros of (1) as a function of z are all real, simple (see Gray and Mathews[2]) and infinite in number, and these results can be extended to the z zeros of (2) (see Cochran[3]). Furthermore, since both (1) and (2) are unaffected by replacing either z by $-z$ or p by $-p$, attention need only be addressed to the case of positive values.

As pointed out by Kline[4] and Waldron[5] the solutions of equations (1) and (2) approach those of the equations $J_p(kz) = 0$ and $J_p'(kz) = 0$, respectively, with increasing k or p. The latter author even indicates the regions among his tabulated values in which this approximation may be reasonably made. Moreover, the familiar asymptotic expressions of McMahon[6] suffice for the calculation of the roots of both (1) and (2) whenever the quantity $\beta = S\pi/(k-1)$ is appreciable, where S is the number of the root when arranged in order of magnitude. As cogently discussed by Waldron,[5] it is convenient to index the roots of the primed equation (2) beginning with $S=0$ rather than with $S=1$ as one does for the solutions of (1). This not only obviates the difficulty wherein, under the usual numbering scheme, the McMahon expression with $\beta = S\pi/(k-1)$ gives the asymptotic expansion for the $(S+1)$st root of (2), but it also serves to set apart the fundamentally different group of roots corresponding to $S=0$. When $p=0$ these special zeros of (2) do not occur; on the other hand, for $p>0$ a representation in terms of powers of $(k-1)/\sqrt{4k}$ has been derived for them by Buchholz.[7]

Formulas

Let z and z' denote roots of the unprimed equation (1) and of the primed equation (2), respectively, and let δ be a positive constant. If $\delta \equiv (k-1)z$ or $\delta \equiv (k-1)z'$, the author has recently developed asymptotic expressions for the Sth zeros of (1) and (2) in the following form:

$$\left\{\begin{matrix} z_{p,S} \\ z'_{p,S} \end{matrix}\right\} = \frac{p\delta}{\sqrt{\delta^2 - (S\pi)^2}} - \frac{\delta}{2} + \frac{1}{p}\left\{\begin{matrix} a(\delta, S) \\ b(\delta, S) \end{matrix}\right\} + 0(p^{-2}). \qquad (3)$$

The functions $a(\delta, S)$ and $b(\delta, S)$, whose precise nature need not concern us here, are independent of p. Solving for $z_{p,S}$ or $z'_{p,S}$ using the first two terms of the expansion yields

$$\left\{\begin{matrix} z_{p,S} \\ z'_{p,S} \end{matrix}\right\} \approx \sqrt{\frac{(S\pi)^2}{(k-1)^2} + \frac{4p^2}{(k+1)^2}} \qquad (4)$$

and

$$z'_{p,0} \approx 2p/(k+1).$$

* Received June 27, 1963.

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